

Auszug aus:

Heath T.L.: The Works of Archimedes, University Press, Cambridge, 1897.



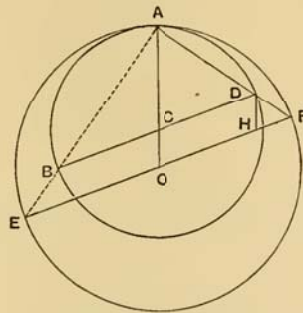
## BOOK OF LEMMAS.

### Proposition 1.

*If two circles touch at  $A$ , and if  $BD$ ,  $EF$  be parallel diameters in them,  $ADF$  is a straight line.*

[The proof in the text only applies to the particular case where the diameters are perpendicular to the radius to the point of contact, but it is easily adapted to the more general case by one small change only.]

Let  $O$ ,  $C$  be the centres of the circles, and let  $OC$  be joined and produced to  $A$ . Draw  $DH$  parallel to  $AO$  meeting  $OF$  in  $H$ .



Then, since  $OH = CD = CA$ ,  
and  $OF = OA$ ,  
we have, by subtraction,

$$HF = CO = DH.$$

Therefore  $\angle HDF = \angle HFD$ .

Thus both the triangles  $CAD$ ,  $HDF$  are isosceles, and the third angles  $ACD$ ,  $DHF$  in each are equal. Therefore the equal angles in each are equal to one another, and

$$\angle ADC = \angle DFH.$$

Add to each the angle  $CDF$ , and it follows that

$$\begin{aligned} \angle ADC + \angle CDF &= \angle CDF + \angle DFH \\ &= (\text{two right angles}). \end{aligned}$$

Hence  $ADF$  is a straight line.

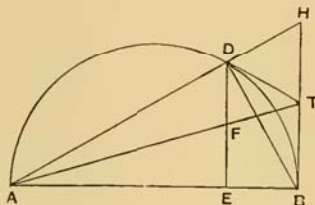
The same proof applies if the circles touch externally\*.

### Proposition 2.

Let  $AB$  be the diameter of a semicircle, and let the tangents to it at  $B$  and at any other point  $D$  on it meet in  $T$ . If now  $DE$  be drawn perpendicular to  $AB$ , and if  $AT$ ,  $DE$  meet in  $F$ ,

$$DF = FE.$$

Produce  $AD$  to meet  $BT$  produced in  $H$ . Then the angle  $ADB$  in the semicircle is right; therefore the angle  $BDH$  is also right. And  $TB$ ,  $TD$  are equal.



Therefore  $T$  is the centre of the semicircle on  $BH$  as diameter, which passes through  $D$ .

Hence  $HT = TB$ .

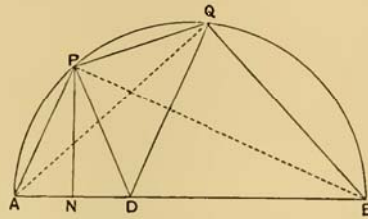
And, since  $DE$ ,  $HB$  are parallel, it follows that  $DF = FE$ .

\* Pappus assumes the result of this proposition in connexion with the  $\alpha\rho\beta\eta\lambda\omicron\varsigma$  (p. 214, ed. Hultsch), and he proves it for the case where the circles touch externally (p. 840).

**Proposition 3.**

Let  $P$  be any point on a segment of a circle whose base is  $AB$ , and let  $PN$  be perpendicular to  $AB$ . Take  $D$  on  $AB$  so that  $AN = ND$ . If now  $PQ$  be an arc equal to the arc  $PA$ , and  $BQ$  be joined,

$BQ, BD$  shall be equal\*.



Join  $PA, PQ, PD, DQ$ .

\* The segment in the figure of the *us.* appears to have been a semicircle, though the proposition is equally true of any segment. But the case where the segment is a semicircle brings the proposition into close connexion with a proposition in Ptolemy's *μεγάλη σύνταξις*, I. 9 (p. 31, ed. Halma; cf. the reproduction in Cantor's *Gesch. d. Mathematik*, I. (1894), p. 389). Ptolemy's object is to connect by an equation the lengths of the chord of an arc and the chord of half the arc. Substantially his procedure is as follows. Suppose  $AP, PQ$  to be equal arcs,  $AB$  the diameter through  $A$ ; and let  $AP, PQ, AQ, PB, QB$  be joined. Measure  $BD$  along  $BA$  equal to  $BQ$ . The perpendicular  $PN$  is now drawn, and it is proved that  $PA = PD$ , and  $AN = ND$ .

$$\text{Then } AN = \frac{1}{2}(BA - BD) = \frac{1}{2}(BA - BQ) = \frac{1}{2}(BA - \sqrt{BA^2 - AQ^2}).$$

And, by similar triangles,  $AN : AP = AP : AB$ .

$$\begin{aligned} \text{Therefore } AP^2 &= AB \cdot AN \\ &= \frac{1}{2}(AB - \sqrt{AB^2 - AQ^2}) \cdot AB. \end{aligned}$$

This gives  $AP$  in terms of  $AQ$  and the known diameter  $AB$ . If we divide by  $AB^2$  throughout, it is seen at once that the proposition gives a geometrical proof of the formula

$$\sin^2 \frac{\alpha}{2} = \frac{1}{2}(1 - \cos \alpha).$$

The case where the segment is a semicircle recalls also the method used by Archimedes at the beginning of the second part of Prop. 3 of the *Measurement of a circle*. It is there proved that, in the figure above,

$$AB + BQ : AC = BP : PA,$$

or, if we divide the first two terms of the proposition by  $AB$ ,

$$(1 + \cos \alpha) / \sin \alpha = \cot \frac{\alpha}{2}.$$

Then, since the arcs  $PA$ ,  $PQ$  are equal,

$$PA = PQ.$$

But, since  $AN = ND$ , and the angles at  $N$  are right,

$$PA = PD.$$

Therefore  $PQ = PD$ ,

and  $\angle PQD = \angle PDQ$ .

Now, since  $A$ ,  $P$ ,  $Q$ ,  $B$  are concyclic,

$$\angle PAD + \angle PQB = (\text{two right angles}),$$

whence  $\angle PDA + \angle PQB = (\text{two right angles})$   
 $= \angle PDA + \angle PDB$ .

Therefore  $\angle PQB = \angle PDB$ ;

and, since the parts, the angles  $PQD$ ,  $PDQ$ , are equal,

$$\angle BQD = \angle BDQ,$$

and  $BQ = BD$ .

#### Proposition 4.

*If  $AB$  be the diameter of a semicircle and  $N$  any point on  $AB$ , and if semicircles be described within the first semicircle and having  $AN$ ,  $BN$  as diameters respectively, the figure included between the circumferences of the three semicircles is "what Archimedes called an ἀρβηλος\*"; and its area is equal to the circle on  $PN$  as diameter, where  $PN$  is perpendicular to  $AB$  and meets the original semicircle in  $P$ .*

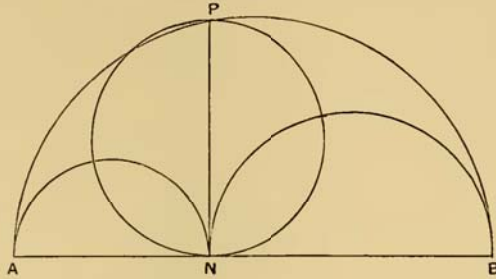
$$\begin{aligned} \text{For } AB^2 &= AN^2 + NB^2 + 2AN \cdot NB \\ &= AN^2 + NB^2 + 2PN^2. \end{aligned}$$

But circles (or semicircles) are to one another as the squares of their radii (or diameters).

\* ἀρβηλος is literally 'a shoemaker's knife.' Cf. note attached to the remarks on the *Liber Assumptorum* in the Introduction, Chapter II.

Hence

$$\begin{aligned} (\text{semicircle on } AB) &= (\text{sum of semicircles on } AN, NB) \\ &+ 2 (\text{semicircle on } PN). \end{aligned}$$



That is, the circle on  $PN$  as diameter is equal to the difference between the semicircle on  $AB$  and the sum of the semicircles on  $AN, NB$ , i.e. is equal to the area of the  $\alpha\rho\beta\eta\lambda\sigma$ .

**Proposition 5.**

*Let  $AB$  be the diameter of a semicircle,  $C$  any point on  $AB$ , and  $CD$  perpendicular to it, and let semicircles be described within the first semicircle and having  $AC, CB$  as diameters. Then, if two circles be drawn touching  $CD$  on different sides and each touching two of the semicircles, the circles so drawn will be equal.*

Let one of the circles touch  $CD$  at  $E$ , the semicircle on  $AB$  in  $F$ , and the semicircle on  $AC$  in  $G$ .

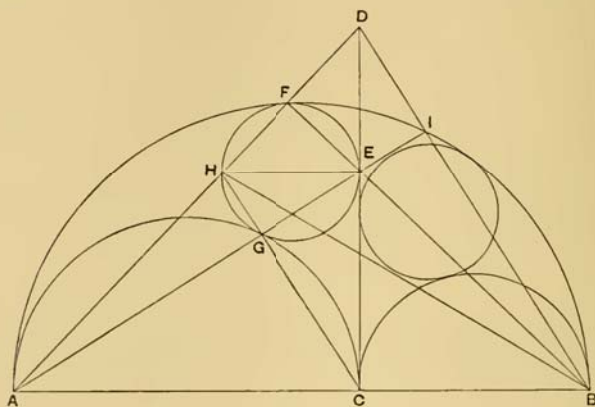
Draw the diameter  $EH$  of the circle, which will accordingly be perpendicular to  $CD$  and therefore parallel to  $AB$ .

Join  $FH, HA$ , and  $FE, EB$ . Then, by Prop. 1,  $FHA, FEB$  are both straight lines, since  $EH, AB$  are parallel.

For the same reason  $AGE, CGH$  are straight lines.

Let  $AF$  produced meet  $CD$  in  $D$ , and let  $AE$  produced meet the outer semicircle in  $I$ . Join  $BI, ID$ .

Then, since the angles  $AFB$ ,  $ACD$  are right, the straight lines  $AD$ ,  $AB$  are such that the perpendiculars on each from the extremity of the other meet in the point  $E$ . Therefore, by the properties of triangles,  $AE$  is perpendicular to the line joining  $B$  to  $D$ .



But  $AE$  is perpendicular to  $BI$ .

Therefore  $BID$  is a straight line.

Now, since the angles at  $G$ ,  $I$  are right,  $CH$  is parallel to  $BD$ .

$$\begin{aligned} \text{Therefore} \quad AB : BC &= AD : DH \\ &= AC : HE, \end{aligned}$$

so that  $AC \cdot CB = AB \cdot HE$ .

In like manner, if  $d$  is the diameter of the other circle, we can prove that  $AC \cdot CB = AB \cdot d$ .

Therefore  $d = HE$ , and the circles are equal\*.

\* The property upon which this result depends, viz. that

$$AB : BC = AC : HE,$$

appears as an intermediate step in a proposition of Pappus (p. 230, ed. Hultsch) which proves that, in the figure above,

$$AB : BC = CE^2 : HE^2.$$

The truth of the latter proposition is easily seen. For, since the angle  $CEH$  is a right angle, and  $EG$  is perpendicular to  $CH$ ,

$$\begin{aligned} CE^2 : EH^2 &= CG : GH \\ &= AC : HE. \end{aligned}$$

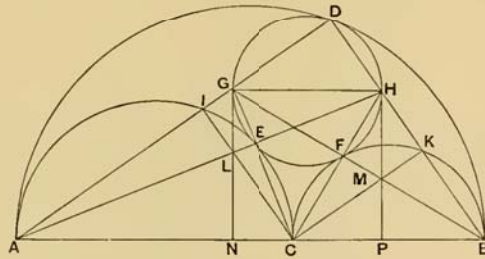
[As pointed out by an Arabian Scholiast Alkauhi, this proposition may be stated more generally. If, instead of one point  $C$  on  $AB$ , we have two points  $C, D$ , and semicircles be described on  $AC, BD$  as diameters, and if, instead of the perpendicular to  $AB$  through  $C$ , we take the radical axis of the two semicircles, then the circles described on different sides of the radical axis and each touching it as well as two of the semicircles are equal. The proof is similar and presents no difficulty.]

**Proposition 6.**

Let  $AB$ , the diameter of a semicircle, be divided at  $C$  so that  $AC = \frac{2}{3} CB$  [or in any ratio]. Describe semicircles within the first semicircle and on  $AC, CB$  as diameters, and suppose a circle drawn touching all three semicircles. If  $GH$  be the diameter of this circle, to find the relation between  $GH$  and  $AB$ .

Let  $GH$  be that diameter of the circle which is parallel to  $AB$ , and let the circle touch the semicircles on  $AB, AC, CB$  in  $D, E, F$  respectively.

Join  $AG, GD$  and  $BH, HD$ . Then, by Prop. 1,  $AGD, BHD$  are straight lines.



For a like reason  $AEH, BFG$  are straight lines, as also are  $CEG, CFH$ .

Let  $AD$  meet the semicircle on  $AC$  in  $I$ , and let  $BD$  meet the semicircle on  $CB$  in  $K$ . Join  $CI, CK$  meeting  $AE, BF$

respectively in  $L, M$ , and let  $GL, HM$  produced meet  $AB$  in  $N, P$  respectively.

Now, in the triangle  $AGC$ , the perpendiculars from  $A, C$  on the opposite sides meet in  $L$ . Therefore, by the properties of triangles,  $GLN$  is perpendicular to  $AC$ .

Similarly  $HMP$  is perpendicular to  $CB$ .

Again, since the angles at  $I, K, D$  are right,  $CK$  is parallel to  $AD$ , and  $CI$  to  $BD$ .

$$\begin{aligned} \text{Therefore} \quad AC : CB &= AL : LH \\ &= AN : NP, \end{aligned}$$

$$\begin{aligned} \text{and} \quad BC : CA &= BM : MG \\ &= BP : PN. \end{aligned}$$

Hence  $AN : NP = NP : PB$ ,  
or  $AN, NP, PB$  are in continued proportion\*.

Now, in the case where  $AC = \frac{3}{2} CB$ ,

$$AN = \frac{3}{2} NP = \frac{9}{4} PB,$$

whence  $BP : PN : NA : AB = 4 : 6 : 9 : 19$ .

$$\text{Therefore} \quad GH = NP = \frac{6}{19} AB.$$

And similarly  $GH$  can be found when  $AC : CB$  is equal to any other given ratio†.

\* This same property appears incidentally in Pappus (p. 226) as an intermediate step in the proof of the "ancient proposition" alluded to below.

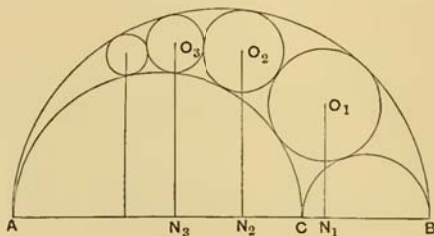
† In general, if  $AC : CB = \lambda : 1$ , we have

$$BP : PN : NA : AB = 1 : \lambda : \lambda^2 : (1 + \lambda + \lambda^2),$$

and

$$GH : AB = \lambda : (1 + \lambda + \lambda^2).$$

It may be interesting to add the enunciation of the "ancient proposition" stated by Pappus (p. 208) and proved by him after several auxiliary lemmas.



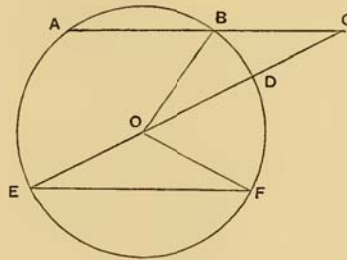
**Proposition 7.**

*If circles be circumscribed about and inscribed in a square, the circumscribed circle is double of the inscribed circle.*

For the ratio of the circumscribed to the inscribed circle is equal to that of the square on the diagonal to the square itself, i.e. to the ratio 2 : 1.

**Proposition 8.**

*If AB be any chord of a circle whose centre is O, and if AB be produced to C so that BC is equal to the radius; if further CO meet the circle in D and be produced to meet the circle a second time in E, the arc AE will be equal to three times the arc BD.*



Draw the chord  $EF$  parallel to  $AB$ , and join  $OB$ ,  $OF$ .

Let an ἀρβηλος be formed by three semicircles on  $AB$ ,  $AC$ ,  $CB$  as diameters, and let a series of circles be described, the first of which touches all three semicircles, while the second touches the first and two of the semicircles forming one end of the ἀρβηλος, the third touches the second and the same two semicircles, and so on. Let the diameters of the successive circles be  $d_1, d_2, d_3, \dots$  their centres  $O_1, O_2, O_3, \dots$  and  $O_1N_1, O_2N_2, O_3N_3, \dots$  the perpendiculars from the centres on  $AB$ . Then it is to be proved that

$$\begin{aligned} O_1N_1 &= d_1, \\ O_2N_2 &= 2d_2, \\ O_3N_3 &= 3d_3, \\ &\dots\dots\dots \\ O_nN_n &= nd_n. \end{aligned}$$

Then, since the angles  $OEF$ ,  $OFE$  are equal,

$$\begin{aligned}\angle COF &= 2 \angle OEF \\ &= 2 \angle BCO, \text{ by parallels,} \\ &= 2 \angle BOD, \text{ since } BC = BO.\end{aligned}$$

Therefore

$$\angle BOF = 3 \angle BOD,$$

so that the arc  $BF$  is equal to three times the arc  $BD$ .

Hence the arc  $AE$ , which is equal to the arc  $BF$ , is equal to three times the arc  $BD$ \*.

**Proposition 9.**

*If in a circle two chords  $AB$ ,  $CD$  which do not pass through the centre intersect at right angles, then*

$$(\text{arc } AD) + (\text{arc } CB) = (\text{arc } AC) + (\text{arc } DB).$$

Let the chords intersect at  $O$ , and draw the diameter  $EF$  parallel to  $AB$  intersecting  $CD$  in  $H$ .  $EF$  will thus bisect  $CD$  at right angles in  $H$ , and

$$(\text{arc } ED) = (\text{arc } EC).$$

Also  $EDF$ ,  $ECF$  are semicircles, while

$$(\text{arc } ED) = (\text{arc } EA) + (\text{arc } AD).$$

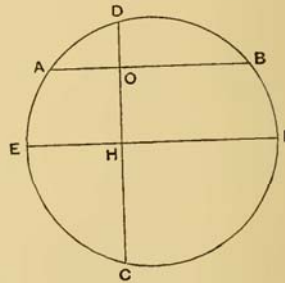
Therefore

$$(\text{sum of arcs } CF, EA, AD) = (\text{arc of a semicircle}).$$

And the arcs  $AE$ ,  $BF$  are equal.

Therefore

$$(\text{arc } CB) + (\text{arc } AD) = (\text{arc of a semicircle}).$$



\* This proposition gives a method of reducing the trisection of any angle, i.e. of any circular arc, to a problem of the kind known as *neûseis*. Suppose that  $AE$  is the arc to be trisected, and that  $ED$  is the diameter through  $E$  of the circle of which  $AE$  is an arc. In order then to find an arc equal to one-third of  $AE$ , we have only to draw through  $A$  a line  $ABC$ , meeting the circle again in  $B$  and  $ED$  produced in  $C$ , such that  $BC$  is equal to the radius of the circle. For a discussion of this and other *neûseis* see the Introduction, Chapter V.

Hence the remainder of the circumference, the sum of the arcs  $AC, DB$ , is also equal to a semicircle; and the proposition is proved.

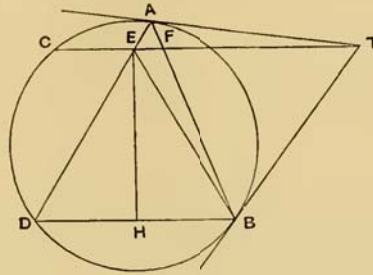
**Proposition 10.**

Suppose that  $TA, TB$  are two tangents to a circle, while  $TC$  cuts it. Let  $BD$  be the chord through  $B$  parallel to  $TC$ , and let  $AD$  meet  $TC$  in  $E$ . Then, if  $EH$  be drawn perpendicular to  $BD$ , it will bisect it in  $H$ .

Let  $AB$  meet  $TC$  in  $F$ , and join  $BE$ .

Now the angle  $TAB$  is equal to the angle in the alternate segment, i.e.

$$\begin{aligned} \angle TAB &= \angle ADB \\ &= \angle AET, \text{ by parallels.} \end{aligned}$$



Hence the triangles  $EAT, AFT$  have one angle equal and another (at  $T$ ) common. They are therefore similar, and

$$FT : AT = AT : ET.$$

Therefore

$$\begin{aligned} ET \cdot TF &= TA^2 \\ &= TB^2. \end{aligned}$$

It follows that the triangles  $EBT, BFT$  are similar.

Therefore 
$$\begin{aligned} \angle TEB &= \angle TBF \\ &= \angle TAB. \end{aligned}$$

But the angle  $TEB$  is equal to the angle  $EBD$ , and the angle  $TAB$  was proved equal to the angle  $EDB$ .

Therefore  $\angle EDB = \angle EBD$ .

And the angles at  $H$  are right angles.

It follows that  $BH = HD^*$ .

**Proposition 11.**

*If two chords  $AB, CD$  in a circle intersect at right angles in a point  $O$ , not being the centre, then*

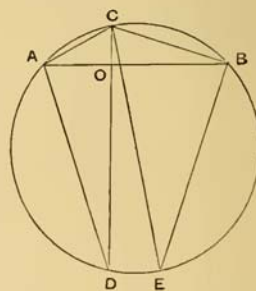
$$AO^2 + BO^2 + CO^2 + DO^2 = (\text{diameter})^2.$$

Draw the diameter  $CE$ , and join  $AC, CB, AD, BE$ .

Then the angle  $CAO$  is equal to the angle  $CEB$  in the same segment, and the angles  $AOC, EBC$  are right; therefore the triangles  $AOC, EBC$  are similar, and

$$\angle ACO = \angle ECB.$$

It follows that the subtended arcs, and therefore the chords  $AD, BE$ , are equal.



\* The figure of this proposition curiously recalls the figure of a problem given by Pappus (pp. 836-8) among his lemmas to the first Book of the treatise of Apollonius *On Contacts* (*περὶ ἐπαφῶν*). The problem is, *Given a circle and two points  $E, F$  (neither of which is necessarily, as in this case, the middle point of the chord of the circle drawn through  $E, F$ ), to draw through  $E, F$  respectively two chords  $AD, AB$  having a common extremity  $A$  and such that  $DB$  is parallel to  $EF$ .* The analysis is as follows. Suppose the problem solved,  $BD$  being parallel to  $FE$ . Let  $BT$ , the tangent at  $B$ , meet  $EF$  produced in  $T$ . ( $T$  is not in general the pole of  $AB$ , so that  $TA$  is not generally the tangent at  $A$ .)

Then  $\angle TBF = \angle BDA$ , in the alternate segment,  
 $= \angle AET$ , by parallels.

Therefore  $A, E, B, T$  are concyclic, and

$$EF \cdot FT = AF \cdot FB.$$

But, the circle  $ADB$  and the point  $F$  being given, the rectangle  $AF \cdot FB$  is given. Also  $EF$  is given.

Hence  $FT$  is known.

Thus, to make the construction, we have only to find the length of  $FT$  from the data, produce  $EF$  to  $T$  so that  $FT$  has the ascertained length, draw the tangent  $TB$ , and then draw  $BD$  parallel to  $EF$ .  $DE, BF$  will then meet in  $A$  on the circle and will be the chords required.

Thus

$$\begin{aligned} (AO^2 + DO^2) + (BO^2 + CO^2) &= AD^2 + BC^2 \\ &= BE^2 + BC^2 \\ &= CE^2. \end{aligned}$$

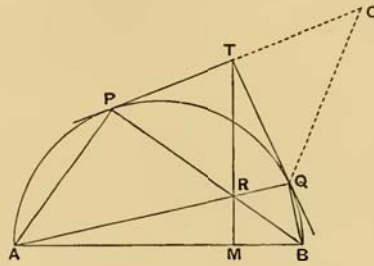
**Proposition 12.**

*If AB be the diameter of a semicircle, and TP, TQ the tangents to it from any point T, and if AQ, BP be joined meeting in R, then TR is perpendicular to AB.*

Let TR produced meet AB in M, and join PA, QB.

Since the angle APB is right,

$$\begin{aligned} \angle PAB + \angle PBA &= (\text{a right angle}) \\ &= \angle AQB. \end{aligned}$$



Add to each side the angle RBQ, and

$$\angle PAB + \angle QBA = (\text{exterior}) \angle PRQ.$$

But  $\angle TPR = \angle PAB$ , and  $\angle TQR = \angle QBA$ ,

in the alternate segments;

therefore  $\angle TPR + \angle TQR = \angle PRQ$ .

It follows from this that  $TP = TQ = TR$ .

[For, if PT be produced to O so that  $TO = TQ$ , we have

$$\angle TOQ = \angle TQO.$$

And, by hypothesis,  $\angle PRQ = \angle TPR + \angle TQR$ .

By addition,  $\angle POQ + \angle PRQ = \angle TPR + \angle OQR$ .

It follows that, in the quadrilateral  $OPRQ$ , the opposite angles are together equal to two right angles. Therefore a circle will go round  $OPQR$ , and  $T$  is its centre, because  $TP = TO = TQ$ . Therefore  $TR = TP$ .]

Thus  $\angle TRP = \angle TPR = \angle PAM$ .

Adding to each the angle  $PRM$ ,

$$\begin{aligned}\angle PAM + \angle PRM &= \angle TRP + \angle PRM \\ &= (\text{two right angles}).\end{aligned}$$

Therefore  $\angle APR + \angle AMR = (\text{two right angles})$ ,

whence  $\angle AMR = (\text{a right angle})^*$ .

### Proposition 13.

If a diameter  $AB$  of a circle meet any chord  $CD$ , not a diameter, in  $E$ , and if  $AM$ ,  $BN$  be drawn perpendicular to  $CD$ , then

$$CN = DM^\dagger.$$

Let  $O$  be the centre of the circle, and  $OH$  perpendicular to  $CD$ . Join  $BM$ , and produce  $HO$  to meet  $BM$  in  $K$ .

Then  $CH = HD$ .

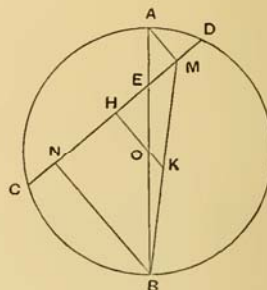
And, by parallels,

since  $BO = OA$ ,

$$BK = KM.$$

Therefore  $NH = HM$ .

Accordingly  $CN = DM$ .



\*  $TM$  is of course the polar of the intersection of  $PQ$ ,  $AB$ , as it is the line joining the poles of  $PQ$ ,  $AB$  respectively.

† This proposition is of course true whether  $M$ ,  $N$  lie on  $CD$  or on  $CD$  produced each way. Pappus proves it for the latter case in his first lemma (p. 788) to the second Book of Apollonius' *νεύσεις*.

**Proposition 14.**

Let  $ACB$  be a semicircle on  $AB$  as diameter, and let  $AD$ ,  $BE$  be equal lengths measured along  $AB$  from  $A$ ,  $B$  respectively. On  $AD$ ,  $BE$  as diameters describe semicircles on the side towards  $C$ , and on  $DE$  as diameter a semicircle on the opposite side. Let the perpendicular to  $AB$  through  $O$ , the centre of the first semicircle, meet the opposite semicircles in  $G$ ,  $F$  respectively.

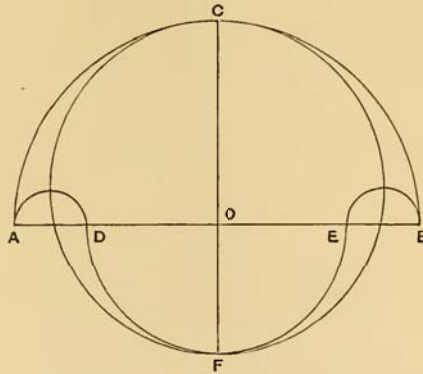
Then shall the area of the figure bounded by the circumferences of all the semicircles ("which Archimedes calls 'Salinon'")\* be equal to the area of the circle on  $CF$  as diameter†.

By Eucl. II. 10, since  $ED$  is bisected at  $O$  and produced to  $A$ ,

$$EA^2 + AD^2 = 2(EO^2 + OA^2),$$

and

$$CF = OA + OE = EA.$$



\* For the explanation of this name see note attached to the remarks on the *Liber Assumptorum* in the Introduction, Chapter II. On the grounds there given at length I believe  $\sigma\lambda\iota\nu\omicron\nu$  to be simply a Graecised form of the Latin word *salinum*, 'salt-cellar.'

† Cantor (*Gesch. d. Mathematik*, I. p. 285) compares this proposition with Hippocrates' attempt to square the circle by means of lunes, but points out that the object of Archimedes may have been the converse of that of Hippocrates. For, whereas Hippocrates wished to find the area of a circle from that of other figures of the same sort, Archimedes' intention was possibly to equate the area of figures bounded by different curves to that of a circle regarded as already known.

Therefore

$$AB^2 + DE^2 = 4(EO^2 + OA^2) = 2(CF^2 + AD^2).$$

But circles (and therefore semicircles) are to one another as the squares on their radii (or diameters).

Therefore

$$\begin{aligned} & (\text{sum of semicircles on } AB, DE) \\ &= (\text{circle on } CF) + (\text{sum of semicircles on } AD, BE). \end{aligned}$$

Therefore

$$(\text{area of 'salinon'}) = (\text{area of circle on } CF \text{ as diam.}).$$

### Proposition 15.

Let  $AB$  be the diameter of a circle,  $AC$  a side of an inscribed regular pentagon,  $D$  the middle point of the arc  $AC$ . Join  $CD$  and produce it to meet  $BA$  produced in  $E$ ; join  $AC$ ,  $DB$  meeting in  $F$ , and draw  $FM$  perpendicular to  $AB$ . Then

$$EM = (\text{radius of circle})^*.$$

Let  $O$  be the centre of the circle, and join  $DA$ ,  $DM$ ,  $DO$ ,  $CB$ .

Now  $\angle ABC = \frac{2}{5}$  (right angle),  
and  $\angle ABD = \angle DBC = \frac{1}{5}$  (right angle),  
whence  $\angle ACD = \frac{2}{5}$  (right angle).

\* Pappus gives (p. 418) a proposition almost identical with this among the lemmas required for the comparison of the five regular polyhedra. His enunciation is substantially as follows. If  $DH$  be half the side of a pentagon inscribed in a circle, while  $DH$  is perpendicular to the radius  $OHA$ , and if  $HM$  be made equal to  $AH$ , then  $OA$  is divided at  $M$  in extreme and mean ratio,  $OM$  being the greater segment.

In the course of the proof it is first shown that  $AD$ ,  $DM$ ,  $MO$  are all equal, as in the proposition above.

Then, the triangles  $ODA$ ,  $DAM$  being similar,

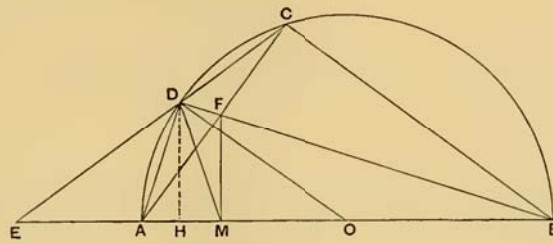
$$OA : AD = AD : AM,$$

or (since  $AD = OM$ )  $OA : OM = OM : MA$ .

Further, the triangles  $FCB$ ,  $FMB$  are equal in all respects.

Therefore, in the triangles  $DCB$ ,  $DMB$ , the sides  $CB$ ,  $MB$  being equal and  $BD$  common, while the angles  $CBD$ ,  $MBD$  are equal,

$$\angle BCD = \angle BMD = \frac{6}{5} \text{ (right angle).}$$



But  $\angle BCD + \angle BAD =$  (two right angles)  
 $= \angle BAD + \angle DAE$   
 $= \angle BMD + \angle DMA,$

so that  $\angle DAE = \angle BCD,$   
 and  $\angle BAD = \angle AMD.$

Therefore  $AD = MD.$

Now, in the triangle  $DMO$ ,

$$\angle MOD = \frac{2}{5} \text{ (right angle),}$$

$$\angle DMO = \frac{6}{5} \text{ (right angle).}$$

Therefore  $\angle ODM = \frac{3}{5} \text{ (right angle) = } \angle AOD;$

whence  $OM = MD.$

Again  $\angle EDA =$  (supplement of  $ADC$ )  
 $= \angle CBA$   
 $= \frac{2}{5} \text{ (right angle)}$   
 $= \angle ODM.$

Therefore, in the triangles  $EDA$ ,  $ODM$ ,

$$\angle EDA = \angle ODM,$$

$$\angle EAD = \angle OMD,$$

and the sides  $AD$ ,  $MD$  are equal.

Hence the triangles are equal in all respects, and

$$EA = MO.$$

Therefore

$$EM = AO.$$

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Moreover  $DE = DO$ ; and it follows that, since  $DE$  is equal to the side of an inscribed hexagon, and  $DC$  is the side of an inscribed decagon,  $EC$  is divided at  $D$  in extreme and mean ratio [i.e.  $EC : ED = ED : DC$ ]; "and this is proved in the book of the Elements." [Eucl. XIII. 9, "If the side of the hexagon and the side of the decagon inscribed in the same circle be put together, the whole straight line is divided in extreme and mean ratio, and the greater segment is the side of the hexagon."]